

Monday, Sep. 23, Lec. 11: Partial derivatives, Gradients

Now: $f: \mathbb{R}^n \rightarrow \mathbb{R}$. Physicists call these "scalar fields".

Many examples.

• Temperature in space at time t .

$$T: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$$

$$T = f(x, y, z, t)$$

• Gravitational potential $U: \mathbb{R}^3 \rightarrow \mathbb{R}$

• Machine learning loss functions:

$$l: \mathbb{R}^{(10^6)} \rightarrow \mathbb{R}.$$

Partial Derivatives

Multiple notations for $z = f(x, y)$ ($f: \mathbb{R}^2 \rightarrow \mathbb{R}$)

$$\frac{\partial z}{\partial x}, z_x, \frac{\partial f}{\partial x}(x, y), D_x f \dots \text{there's a lot.}$$

• These are still functions: $\frac{\partial z}{\partial x}: \mathbb{R}^2 \rightarrow \mathbb{R}$.

If we want the derivative at a point,

$$\left. \frac{\partial z}{\partial x} \right|_{(x, y)}, \text{ or } \frac{\partial z}{\partial x}(x, y)$$

$$\text{Ex: } f(x, y) = \frac{1}{2}(x^2 + y^2)$$

$$\frac{\partial f}{\partial x} = x, \quad \left. \frac{\partial f}{\partial x} \right|_{(0,1)} = 0, \quad f_y(0,1) = y|_{(0,1)} = 1.$$

Change-of-variables

Ex: polar coordinates

$$z = x^2 - y^2, \quad \left. \frac{\partial z}{\partial x} \right|_{(x,y)} = 2x$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$\Rightarrow z(r, \theta) = r^2 (\cos^2 \theta - \sin^2 \theta)$$

$$\frac{\partial z}{\partial r} = 2r (\cos^2 \theta - \sin^2 \theta)$$

Special case: Change-of-basis
is a linear change-of-variables,
S invertible, and

$$[V]_{\text{new}} = S [V]_{\text{old}}$$

Gradient:

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$. Then the gradient of f , $\nabla f: \mathbb{R}^n \rightarrow \mathbb{R}^n$
is the vector field

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

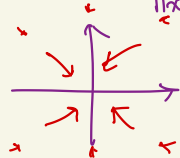
Ex:

Gravitational potential energy $U: \mathbb{R}^3 \rightarrow \mathbb{R}$ (of point mass M Gd)

$$U(\vec{x}) = -\frac{1}{\|\vec{x}\|}$$

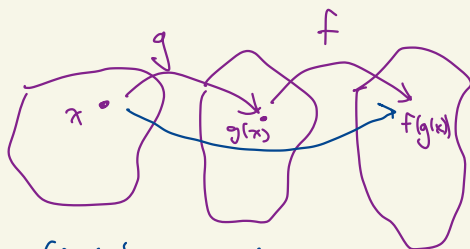
Given a potential, its gradient has the interpretation
of force. Gravitational force: unit vector in x direction

$$\nabla U(\vec{x}) = -\frac{1}{\|\vec{x}\|^3} \vec{x} = -\frac{1}{\|\vec{x}\|^2} \hat{\vec{x}}$$



Chain rule:

Single variable:



$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Multi-variable: Need to track domains and codomains.

Special common case:

$$\gamma: \mathbb{R} \rightarrow \mathbb{R}^n, \quad f: \mathbb{R}^n \rightarrow \mathbb{R}$$

eg. motion
of particle

eg. potential



$$\gamma(t) = (x_1(t), \dots, x_n(t))$$

$$f(x_1, x_2, \dots, x_n) = z.$$

$$f \circ \gamma: \mathbb{R} \rightarrow \mathbb{R}$$

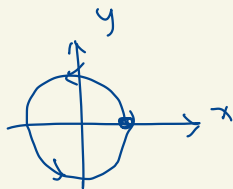
$$\begin{aligned} (f \circ \gamma)'(t) &= \nabla f(\gamma(t)) \cdot \gamma'(t) \\ &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right) \cdot \left(\frac{\partial x_1}{\partial t}, \dots, \frac{\partial x_n}{\partial t} \right) \\ &= \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t} + \dots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial t}. \end{aligned}$$

Ex:

Sps a particle is moving in potential

$$z = x^2 y$$

$$\gamma(t) = (\sin t, \cos t) \\ t \in [0, 2\pi)$$



How fast is z changing?

$$\begin{aligned} \frac{dz}{dt} &= \nabla z \cdot \gamma'(t) = (2xy, x^2) \cdot (\cos t, -\sin t) \\ &= (2 \sin t \cos t, \sin^2 t) \cdot (\cos t, -\sin t) \\ &= 2 \sin t \cos^2 t - \sin^3 t. \end{aligned}$$

$$\begin{aligned} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = 2xy \cdot (\cos t) + x^2 (-\sin t) \\ &\quad \begin{matrix} x = x(t) \\ y = y(t) \end{matrix} = 2 \sin t \cos^2 t - \sin^3 t. \end{aligned}$$

Differentials

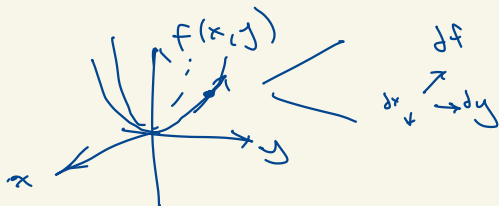
Very common to instead describe "little changes" in a function, rather than rates of change:

if $f: \mathbb{R}^n \rightarrow \mathbb{R}$,

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

Ex:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$



Physics note:

- One nice conceptual thing: if x is length,

$$\frac{\partial}{\partial x} \text{ has units } \left[\frac{1}{\text{length}} \right], \quad dx \text{ has units } [\text{length}]$$

$$\frac{1}{dx}$$

And similarly for other units.

Ex:

$$\text{Let } f(x, y) = x^2 + y^2$$

Suppose we increase $x \mapsto x + \epsilon$ and $y \mapsto y + \epsilon$. How does f change?

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 2x dx + 2y dy.$$

$$\text{So } \Delta x = \epsilon, \Delta y = \epsilon,$$

$$\Delta f = 2x\epsilon + 2y\epsilon$$

Compare:

$$\begin{aligned} \Delta f &= f(x+\epsilon, y+\epsilon) - f(x, y) = (x+\epsilon)^2 + (y+\epsilon)^2 - (x^2 + y^2) \\ &= x^2 + 2x\epsilon + \epsilon^2 + y^2 + 2y\epsilon + \epsilon^2 - x^2 - y^2 \\ &= 2x\epsilon + 2y\epsilon + O(\epsilon)^2 \end{aligned}$$

Implicit Differentiation

Ex: Given $x + e^x = t$, find $\frac{dx}{dt}$.

Soln:

$$\frac{dx}{dt} + e^x \cdot \frac{dx}{dt} = \frac{dt}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{1+e^x}$$

Alt:

$$d(x + e^x) = dt$$

$$dx + e^x dx$$

$$\Rightarrow (1 + e^x) dx = dt$$

$$\frac{dx}{dt} = \frac{1}{1+e^x}$$

Ex: Find tangent line to curve $x^3 - 3y^3 + xy + 2 = 0$ at $(1, 2)$.

Soln:

$$\begin{aligned} 0 &= \frac{d}{dx}(\quad) = 3x^2 \frac{dx}{dx} - 9y^2 \frac{dy}{dx} + \frac{dx}{dx} y + x \frac{dy}{dx} + 0 \\ &= 3x^2 - 9y^2 \frac{dy}{dx} + y + x \frac{dy}{dx} \end{aligned}$$

$$\text{at } (x, y) = (1, 2),$$

$$0 = 3 - 9(4) \frac{dy}{dx} + 2 + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{7} \quad \text{Then, } y = mx + b, \text{ so } y = \frac{1}{7}x + b.$$

$$2 = \frac{1}{7}(1) + b \Rightarrow b = 2 - \frac{1}{7}$$

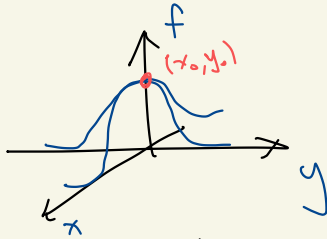
⇒ Tangent line is

$$y = \frac{1}{7}x + 2 - \frac{1}{7}$$

Optimization

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$ tells us everything we need.

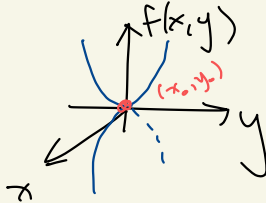
Max:



In all cases,

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right) = 0.$$

Saddle:



Min

